Proof Answers

Q. no.	A nec. or suff. for B?	Statement A	$\Rightarrow$, $\Leftarrow$, $\Leftrightarrow$, 'none'?	Statement B	B nec. or suff. for A?
1	S	$x^2 = 9$	$\Rightarrow$	$x < 6$	N
2	N, S	This month has 29 days	$\Leftrightarrow$	It is February in a leap year	N, S
3	N, S	$x = +1$ or $x = -1$	$\Leftrightarrow$	$(x+1)(x-1) = 0$	N, S
4	N	$\sin x = 1$	$\Leftarrow$	$x = 90°$	S
5	S	A polygon is a square	$\Rightarrow$	A polygon has 4 sides	N
6	N	$(x+1)(y-1) = 0$	$\Leftarrow$	$x = -1$ and $y = +1$	S
7	N, S	Three straight lines in 2d meet in exactly 2 points	$\Leftrightarrow$	Exactly 2 of 3 straight lines in 2d are parallel	N, S
8	N	$x^2 = 9$	$\Leftarrow$	$x = 3$	S
9	N, S	n^2 is a multiple of 5	$\Leftrightarrow$	n is a multiple of 5	N, S
10	neither	k is divisible by 3	'none'	$k + 1$ is even	neither
11	N	The discriminant of a quadratic equation is non-negative	$\Leftarrow$	A quadratic equation has 2 unequal real roots	S
12	neither	$a > b$	'none'	$\frac{1}{a} < \frac{1}{b}$	neither

Chapter 2: Proof

1. Proof

Exercise 1.1 — Proof

Q1 B is true — for example, if $x = -4$, $x^2 > 9$ but $x < 3$, so A and C do not hold.

Q2 **a)** Take two odd numbers $2l + 1$ and $2m + 1$ (where l and m are integers), then their sum is $2l + 1 + 2m + 1 = 2l + 2m + 2 = 2(l + m + 1)$ = even.

b) Take two even numbers, $2j$ and $2k$ (where j and k are integers), then their product is $2j \times 2k = 4jk = 2(2jk)$ = even.

c) Take one even number, $2l$ and one odd number $2m + 1$ (where l and m are integers), then their product is $2l \times (2m + 1) = 4lm + 2l = 2(2lm + l)$ = even.

Q3 E.g. Let $p = 1 \Rightarrow \frac{1}{p^2} = \frac{1}{p}$, so the statement is not true.
You could have also taken p to be a negative number.

Q4
$$\begin{aligned}(x + 5)^2 + 3(x - 1)^2 &= x^2 + 10x + 25 + 3(x^2 - 2x + 1)\\ &= x^2 + 10x + 25 + 3x^2 - 6x + 3\\ &= 4x^2 + 4x + 28\\ &= 4(x^2 + x + 7)\end{aligned}$$

This has a factor of 4 outside the brackets, so it is always divisible by 4.

Q5 Proof by exhaustion:
Take three consecutive integers $(n - 1)$, n and $(n + 1)$.
Their product is $(n - 1)n(n + 1) = n(n^2 - 1) = n^3 - n$.
Consider the two cases — n even and n odd.
For n even, n^3 is even (as even × even = even) so $n^3 - n$ is also even (as even – even = even). For n odd, n^3 is odd (as odd × odd = odd) so $n^3 - n$ is even (as odd – odd = even). So $n^3 - n$ is even when n is even and when n is odd, and n must be either odd or even, so the product of three consecutive integers is always even.
Another approach to this proof is to take the product of three consecutive integers $n(n + 1)(n + 2)$ and consider n odd and n even. If n is odd:
$n(n + 1)(n + 2)$ = (odd × even) × odd = even × odd = even.
If n is even:
$n(n + 1)(n + 2)$ = (even × odd) × even = even × even = even.

Q6 The simplest way to disprove the statement is to find a counter-example. Try some values of n and see if the statement is true for them:

$n = 3 \Rightarrow n^2 - n - 1 = 3^2 - 3 - 1 = 5$ — prime
$n = 4 \Rightarrow n^2 - n - 1 = 4^2 - 4 - 1 = 11$ — prime
$n = 5 \Rightarrow n^2 - n - 1 = 5^2 - 5 - 1 = 19$ — prime
$n = 6 \Rightarrow n^2 - n - 1 = 6^2 - 6 - 1 = 29$ — prime
$n = 7 \Rightarrow n^2 - n - 1 = 7^2 - 7 - 1 = 41$ — prime
$n = 8 \Rightarrow n^2 - n - 1 = 8^2 - 8 - 1 = 55$ — not prime

$n^2 - n - 1$ is not prime when $n = 8$.
So the statement is false.

Sometimes good old trial and error is the easiest way to find a counter-example. Don't forget, if you've been told to disprove a statement like this, then a counter-example must exist.

Q7 Find a counter-example for which the statement isn't true. Take $x = -1$ and $y = 2$. Then
$\sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5} = 2.236...$
and $x + y = -1 + 2 = 1$. $2.236... > 1$,
so the statement is not true.

Q8 Take any two rational numbers a and b. By the definition of rational numbers we know that $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where p, q, r and s are integers,
and q & s are non-zero.
So, the sum of a and b is $\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$.
ps and rq are the product of integers, so are also integers. This means $ps + rq$ is also an integer. qs is the product of non-zero integers, so must also be a non zero integer. This shows that $a + b$ is the quotient of two integers, and has a non-zero denominator, so by definition $a + b$ is rational.

Q9 **a)** Proof by exhaustion:
Consider the two cases — n even and n odd.
Let n be even.
$n^2 - n = n(n - 1)$.
If n is even, $n - 1$ is odd so $n(n - 1)$ is even (as even × odd = even). This means that $n(n - 1) - 1$ is odd.
Let n be odd. If n is odd, $n - 1$ is even, so $n(n - 1)$ is even (as odd × even = even). This means that $n(n - 1) - 1$ is odd.
As any integer n has to be either odd or even, $n^2 - n - 1$ is odd for any value of n.

b) As $n^2 - n - 1$ is odd, $n^2 - n - 2$ is even.
The product of even numbers is also even, so as $(n^2 - n - 2)^3$ is the product of 3 even numbers, it will always be even.